Roots of Polynomial Functions

Fundamental Theorem of Algebra

Given a polynomial function on degree n in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots a_2 x^2 + a_1 x + a_0$$

(with $a_n \neq 0$) has exactly n roots in the complex number system.

This means that if we set f(x)=0, there are exactly n different values of x that will satisfy the resulting equation.

Example:

The equation:

$$f(x) = 3x^3 - 3x^2 - 11x - 6$$

has exactly 3 roots in the complex number system.

Rational Roots Test

A test that will show all *possible* rational roots to any polynomial function.

For the rational roots test, and this polynomial:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots a_2 x^2 + a_1 x + a_0$$

Let p equal all of the positive and negative factors of a_0 and let q equal all of the positive and negative factors of a_n. All of the possible rational roots to f(x) are given as p/q for each factor p and each factor q.

Example:

$$f(x) = 2x^3 + 11x^2 + 17x + 6$$
$$p = \pm 1, \pm 2, \pm 3, \pm 6$$
$$q = \pm 1, \pm 2$$
$$\frac{p}{q} = \frac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2}$$
$$\frac{p}{q} = -1,\ 1,\ -2,\ 2,\ -3,\ 3,\ -6,\ 6,\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{3}{2},\ \frac{3}{2}$$

These are all the possible rational roots of the function above. Notice that there is no need to list 2/2 and 6/2 because they are listed as 1 and 3 from other factors. The above possible roots would be the same is the constant term was -6 since all factors are plus or minus.

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Descarte's Rule of Signs:
A test that will show the possible number of positive real roots and the possible number of negative real roots.

For Descarte's rule of signs, count the sign changes between each term of f(x) and call that r. The number of positive real roots of the function is given by:
$r-2k$

For all whole numbers k such that the number of roots will never be less than 0.

Example:

In this polynomial, there are 2 sign changes show in red.

$f(x)=4x^5-2x^3-11x^2+17x+6$

Since there are 2 sign changes (in red), there are 2 or 0 positive real roots.

The number of negative real roots is the same method, however you use f(-x) instead.

Example:

$f(-x)=4(-x)^5-2(-x)^3-11(-x)^2+17(-x)+6$
$f(-x)=-4x^5+2x^3-11x^2-17x+6$

Since there are 3 sign changes, the function f(x) must have 3 or 1 negative real roots.

NOTE: when finding the number of negative real roots, it is going to be the number of roots for f(x), NOT f(-x).

Complex Roots:
All complex roots of functions come in pairs. If one complex number is a root, then another root will be the complex conjugate of the root.

Example:

If 2+4i is a root of a polynomial function, 2-4i will also be a root of the polynomial function.

Finding Roots:
Any root of a polynomial equation will be able to divide the equation with no remainder when written as a linear factor.

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<u>Example</u>:

$f(x) = 2x^3 + 11x^2 + 17x + 6$

Has exactly 2 roots in the complex number system. If $x = -1$ is a root (shown to be a possible root from the rational roots test above), then $\left(2x^3 + 11x^2 + 17x + 6\right) \div (x+1)$ should not have a remainder.

$\left(2x^3 + 11x^2 + 17x + 6\right) \div (x+1) = \left(2x^2 + 7x + 3\right)$

Since there is no remainder, $x = -1$ is a factor of the given polynomial.

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